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Generalizations of Clausen's formula and algebraic transformations of Calabi-Yau differential equations. (English summary)

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The authors present an unusual generalization of the Clausen and Orr-type theorems [L. J. Slater, *Generalized hypergeometric functions*, Cambridge Univ. Press, Cambridge, 1966; MR0201688 (34 #1570) (§2.5)] for fourth- and fifth-order hypergeometric equations. To describe the authors' main result let α, a, b, c be a generic set of complex parameters and set $\hat{a} = a$, $\hat{b} = a - 2b$ and $\hat{c} = a^2 - 4c$. Further, define the differential operators

$$\widehat{D} := \theta^5 - z(2\theta+1)(\theta+\alpha)(\theta+1-\alpha)(\hat{a}\theta^2+\hat{a}\theta+\hat{b}) \\ + \hat{c}z^2(\theta+1)(\theta+\alpha)(\theta+1-\alpha)(\theta+1+\alpha)(\theta+2+\alpha)$$

and

$$D := \theta^4 - z((4\hat{a}(\theta+\tfrac{1}{2})^4) + ((p+4)\hat{a}-2\hat{b})(\theta+\tfrac{1}{2})^2 \\ + \tfrac{1}{4}(1-p)\hat{a} - \tfrac{1}{2}(1-2p)\hat{b}) + z^2((6\hat{a}^2-8\hat{c})(\theta+1)^4 \\ + (\tfrac{3}{2}(5+2p)\hat{a}^2 - 4\hat{a}\hat{b} - 2(13+2p)\hat{c})(\theta+1)^2 \\ + \tfrac{3}{4}\hat{a}^2 - (1-p)\hat{a}\hat{b} + \hat{b}^2 - (2+2p-p^2)\hat{c}) \\ - (\hat{a}^2-4\hat{c})z^3(\theta+\tfrac{3}{2})^2(4\hat{a}(\theta+\tfrac{3}{2})^2 + 3(1+p)\hat{a} - 2\hat{b}) \\ + (\hat{a}^2-4\hat{c})^2z^4(\theta+\tfrac{3}{2})(\theta+\tfrac{5}{2})(\theta+\tfrac{3}{2}+\alpha)(\theta+\tfrac{5}{2}-\alpha),$$

in which $\theta := z(d/dz)$ and $p = \alpha(1-\alpha)$. The main result obtained by the authors is contained in the following general theorem.

Theorem. Let $\hat{F}(z) \in 1 + z\mathbb{C}[\![z]\!]$ be the analytic solution of the differential equation $Dy = 0$, and let $F(z) \in 1 + z\mathbb{C}[\![z]\!]$ be the Hadamard product of

$$f_\alpha(z) = \frac{1}{1-z} \cdot {}_2F_1\left(\begin{matrix}\alpha, 1-\alpha \\ 1\end{matrix}\,\middle|\, -\frac{z}{1-z}\right)$$

and the analytic solution of the differential equation $\widehat{D}y = 0$. Then

$$F(z) = \frac{1-cz^2}{(1-az+cz^2)^{3/2}}\hat{F}\left(-\frac{-z}{1-az+cz^2}\right).$$

Further, as an application of the derived results, the authors obtain several examples of the algebraic transformations of Calabi-Yau differential equations.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.